



Oscillation criteria for second order nonlinear dynamic equations with impulses

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ABSTRACT

By using an impulsive inequality and the Riccati transformation technique on time scales, several oscillation criteria are established for the second order nonlinear dynamic equations on time scales with impulses. Examples are given to show that the impulses play a dominant part in the oscillations of dynamic equations on time scales.

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1. Introduction

We are concerned with the oscillation of second order nonlinear dynamic equations with impulses

$$\begin{cases} (r(t)(y^\Delta(t))^\alpha)^\Delta + f(t, y^\sigma(t)) = 0, & t \in \mathbb{T} := [0, \infty) \cap \mathbb{T}, t \neq t_k, k = 1, 2, \dots, \\ y(t_k^+) = g_k(y(t_k^-)), y^\Delta(t_k^+) = h_k(y^\Delta(t_k^-)), & k = 1, 2, \dots, \\ y(t_0^+) = y_0, & y^\Delta(t_0^+) = y_0^\Delta, \end{cases} \quad (1.1)$$

where α is the quotient of positive odd integers, \mathbb{T} is an unbounded-above time scale with $0 \in \mathbb{T}$, $t_k \in \mathbb{T}$, $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$, $\lim_{k \rightarrow \infty} t_k = \infty$.

$$y(t_k^+) = \lim_{h \rightarrow 0^+} y(t_k + h), \quad y^\Delta(t_k^+) = \lim_{h \rightarrow 0^+} y^\Delta(t_k + h), \quad (1.2)$$

which represent the right limits of $y(t)$ at $t = t_k$ in the sense of time scales, and in addition, if t_k is right scattered, then $y(t_k^+) = y(t_k)$, $y^\Delta(t_k^+) = y^\Delta(t_k)$. We can define $y(t_k^-)$, $y^\Delta(t_k^-)$ similarly to (1.2).

We always suppose that the following conditions hold:

(H₁) $f \in C_{rd}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $yf(t, y) > 0$ ($y \neq 0$) and $\frac{f(t, y)}{\varphi(y)} \geq p(t)$ ($y \neq 0$), where $p(t) \in C_{rd}(\mathbb{T}, [0, +\infty))$, $\varphi(y) \in C^1(\mathbb{R}, \mathbb{R})$ and $y\varphi(y) > 0$ ($y \neq 0$), $\varphi'(y) \geq 0$.

(H₂) $g_k, h_k \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants a_k, a_k^*, b_k, b_k^* such that

$$a_k^* \leq \frac{g_k(y)}{y} \leq a_k, \quad b_k^* \leq \frac{h_k(y)}{y} \leq b_k, \quad y \neq 0, k = 1, 2, \dots$$

The theory of time scales, which provides powerful new tools for exploring connections between the traditionally separated fields, has been developing rapidly and has received much attention. Dynamic equations can not only unify

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the theories of differential equations and difference equations, but also extend these classical cases to cases “in between” and can be applied to other different types of time scales. The theory of dynamic equations on time scales is an adequate mathematical apparatus for the simulation of processes and phenomena observed in biotechnology, chemical technology, economic, neural networks, physics, social sciences etc. For further applications and questions concerning solutions of dynamic equations on time scales, see [1–3]

In recent years, there has been an increasing interest in studying the existence of solutions, the oscillation and nonoscillation of dynamic equations on time scales, see [4]. The existence of solutions to dynamic equations with impulses, we refer the reader to Agarwal et al. [5], Belarbi et al. [6], Benchohra et al. [7–10], Chang et al. [11] and so forth. In [10], Benchohra et al. considered the existence of extremal solutions for a class of second order impulsive dynamic equations on time scales, we can see that the existence of global solutions can be guaranteed by some simple conditions.

Based on the oscillatory behavior of the impulsive dynamic equations on time scales, Benchohra et al. [7] discuss the existence of oscillatory and nonoscillatory solutions by lower and upper solutions method for the first order impulsive dynamic equations on certain time scales

$$\begin{cases} y^\Delta(t) = f(t, y(t)), & t \in \mathbb{T} := [0, \infty) \cap \mathbb{T}, t \neq t_k, k = 1, 2, \dots, \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, 2, \dots \end{cases} \quad (1.3)$$

Huang et al. [12,13] considered the second order nonlinear impulsive dynamic equations on time scales

$$\begin{cases} y^{\Delta\Delta}(t) + f(t, y^\sigma(t)) = 0, & t \in \mathbb{T} := [0, \infty) \cap \mathbb{T}, t \neq t_k, k = 1, 2, \dots, \\ y(t_k^+) = g_k(y(t_k^-)), & y^\Delta(t_k^+) = h_k(y^\Delta(t_k^-)), & k = 1, 2, \dots, \\ y(t_0^+) = y_0, & y^\Delta(t_0^+) = y_0^\Delta, \end{cases} \quad (1.4)$$

extend the well-known results of Chen [14] and Peng [15] without delay for the impulsive differential equations to (1.4).

Following this trend, to develop the qualitative theory of dynamic equations on time scales with impulses, the following question arises. Can we obtain oscillation criteria on time scales which improve the results established in Huang [12,13] and [16–18], and from which we are able to deduce the corresponding results for differential and difference equations as a special case, cover oscillation criteria of the type established by Chen, Peng and others?

The aim of this paper is to give a positive answer to this question by extending the impulsive inequality and Riccati transformation techniques in a time scale setting to obtain some new oscillation criteria of the Chen–Peng type for Eq. (1.1). Our results in this paper improve the results of Huang [12,13], and can be applied to arbitrary time scales. Some examples are given to show that though a dynamic equation is nonoscillatory, it may become oscillatory by adding some impulses to it. That is, in this cases, impulses play a dominating part in oscillations of dynamic equations on time scales.

Throughout the remainder of the paper, we assume that, for each $k = 1, 2, \dots$, the points of impulses t_k are right-dense (rd for short). In order to define the solutions of the problem (1.1), we introduce the following space

$AC^i = \{y : \mathbb{T} \rightarrow \mathbb{R} \text{ is } i\text{-times } \Delta\text{-differentiable, whose } i\text{th delta derivative } y^{\Delta(i)} \text{ is absolutely continuous}\}.$

$PC = \{y : \mathbb{T} \rightarrow \mathbb{R} \text{ is rd-continuous expect at the points } t_k, k = 1, 2, \dots, \text{ for which } y(t_k^-), y(t_k^+), y^\Delta(t_k^-) \text{ and } y^\Delta(t_k^+) \text{ exist with } y(t_k^-) = y(t_k), y^\Delta(t_k^-) = y^\Delta(t_k)\}.$

Definition 1.1. A function $y \in PC \cap AC^2(\mathbb{T} \setminus \{t_1, t_2, \dots\}, \mathbb{R})$ is said to be a solution of (1.1), if it satisfies $(r(t)(y^\Delta(t))^\alpha)^\Delta + f(t, y^\sigma(t)) = 0$ a.e. on $\mathbb{T} \setminus \{t_k\}, k = 1, 2, \dots$, and for each $k = 1, 2, \dots, y$ satisfies the impulsive condition $y(t_k^+) = g_k(y(t_k^-)), y^\Delta(t_k^+) = h_k(y^\Delta(t_k^-))$ and the initial conditions $y(t_0^+) = y_0, y^\Delta(t_0^+) = y_0^\Delta$.

Definition 1.2. A solution y of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Eq. (1.1) is called oscillatory if all solutions are oscillatory.

2. Some preliminaries

We will briefly recall some basic definitions and facts from the time scale calculus that we will use in the sequel. For more details see [2,3,19].

On any time scale \mathbb{T} , we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$, and \emptyset denotes the empty set. A nonmaximal element $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. A nonminimal element $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and left-scattered if $\rho(t) < t$. The graininess μ of the time scale \mathbb{T} is defined by $\mu(t) = \sigma(t) - t$.

A mapping $f : \mathbb{T} \rightarrow \mathbb{X}$ is said to be differentiable at $t \in \mathbb{T}$, if there exists $f^\Delta(t) \in \mathbb{X}$ such that for any $\varepsilon > 0$, there exists a neighborhood \mathbf{U} of t satisfying $|\{f(\sigma(t)) - f(s)\} - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$, for all $s \in \mathbf{U}$. We say that f is delta differentiable (or in short: differentiable) on \mathbb{T} provided $f^\Delta(t)$ exist for all $t \in \mathbb{T}$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist (finite) at left-dense points in \mathbb{T} . The set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and forward jump operator σ are related by the formula

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t). \quad (2.1)$$

Let f be a differentiable function on $[a, b]_{\mathbb{T}}$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]_{\mathbb{T}}$ if $f^\Delta > 0$, $f^\Delta < 0$, $f^\Delta \geq 0$ and $f^\Delta \leq 0$ for all $t \in [a, b]_{\mathbb{T}}$, respectively.

We will make use of the following product fg and quotient $\frac{f}{g}$ rules for the derivative of two differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \quad (2.2)$$

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}, \quad (2.3)$$

where $f^\sigma = f \circ \sigma$, $gg^\sigma \neq 0$.

The integration by parts formula reads

$$\int_a^b f^\Delta(t)g(t)\Delta t = f(t)g(t)\Big|_a^b - \int_a^b f^\sigma(t)g^\Delta(t)\Delta t. \quad (2.4)$$

Chain Rule: Assume $g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable on \mathbb{T} and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. Then $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable and satisfies

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh \right\} g^\Delta(t). \quad (2.5)$$

A function $p : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive if for all $t \in \mathbb{T}$

$$1 + \mu(t)p(t) \neq 0.$$

The set of all *rd-continuous* function f which satisfy $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$ will be denoted by \mathcal{R}^+ . The generalized exponential function e_p is defined by

$$e_p(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\},$$

with $\xi_h(z) = \frac{\log(1+hz)}{h}$ if $h \neq 0$ and $\xi_h(z) = z$ if $h = 0$.

Lemma 2.1. Let $y, f \in C_{rd}$ and $p \in \mathcal{R}^+$. Then

$$y^\Delta(t) \leq p(t)y(t) + f(t),$$

implies, for all $t \in \mathbb{T}$

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(s))f(s)\Delta s.$$

See [5, P. 255].

3. Main results

In this section, we will employ the Riccati substitution on time scales and establish new oscillation criteria for (1.1). Before we state and prove our main oscillation results, we prove some lemmas which are important in the proof of the main results.

Lemma 3.1. Assume that $m \in PC \cap AC^1(\mathbb{J}_{\mathbb{T}} \setminus \{t_1, t_2, \dots\}, \mathbb{R})$ and

$$\begin{cases} m^\Delta(t) \leq p(t)m(t) + q(t), & t \in \mathbb{J}_{\mathbb{T}} := [0, \infty) \cap \mathbb{T}, t \neq t_k, k = 1, 2, \dots, \\ m(t_k^+) \leq d_k m(t_k) + b_k, & k = 1, 2, \dots, \end{cases} \quad (3.1)$$

then for $t \geq t_0$

$$m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k e_p(t, t_0) + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} d_j e_p(t, t_k) \right) b_k + \int_{t_0}^t \prod_{s < t_k < t} d_k e_p(t, \sigma(s))q(s)\Delta s. \quad (3.2)$$

See the Lemma 3.1 of Huang [13].

Lemma 3.2. Suppose that (H_1) , (H_2) hold and $y(t) > 0$, $t \geq T \geq t_0$ is a nonoscillatory solution of (1.1). If

$$(H_3) \int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)} \prod_{t_0 < t_k < s} \frac{b_k^*}{a_k} \Delta s = \infty,$$

then $y^\Delta(t_k^+) \geq 0$ and $y^\Delta(t) \geq 0$ for $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, where $t_k \geq T$.

Proof. At first, we prove that $y^\Delta(t_k) \geq 0$ for $t_k \geq T$, otherwise, there exists some j such that $t_j \geq T$ and $y^\Delta(t_j) < 0$, hence

$$y^\Delta(t_j^+) = h_j(y^\Delta(t_j)) \leq b_j^* y^\Delta(t_j) < 0.$$

Let $y^\Delta(t_j^+) = -\beta$ ($\beta > 0$). From (1.1) and (H_1) , for $t \in (t_{j+i-1}, t_{j+i}]_{\mathbb{T}}$, $i = 1, 2, \dots$, we obtain

$$(r(t)(y^\Delta(t))^\alpha)^\Delta = -f(t, y^\sigma(t)) \leq -p(t)\varphi(y^\sigma(t)) \leq 0,$$

i.e. $r(t)(y^\Delta(t))^\alpha$ is nonincreasing in $(t_{j+i-1}, t_{j+i}]_{\mathbb{T}}$, $i = 1, 2, \dots$. Since α is the quotient of positive integers, we obtain

$$\begin{aligned} y^\Delta(t_{j+1}) &\leq \left(\frac{r(t_j)}{r(t_{j+1})} \right)^{\frac{1}{\alpha}} y^\Delta(t_j^+) = -\beta \left(\frac{r(t_j)}{r(t_{j+1})} \right)^{\frac{1}{\alpha}} < 0, \\ y^\Delta(t_{j+2}) &\leq \left(\frac{r(t_{j+1})}{r(t_{j+2})} \right)^{\frac{1}{\alpha}} y^\Delta(t_{j+1}^+) = \left(\frac{r(t_{j+1})}{r(t_{j+2})} \right)^{\frac{1}{\alpha}} h_{j+1}(y^\Delta(t_{j+1})) \\ &\leq b_{j+1}^* \left(\frac{r(t_{j+1})}{r(t_{j+2})} \right)^{\frac{1}{\alpha}} y^\Delta(t_{j+1}) \leq -b_{j+1}^* \beta \left(\frac{r(t_j)}{r(t_{j+2})} \right)^{\frac{1}{\alpha}} < 0. \end{aligned} \quad (3.3)$$

By induction, we obtain

$$y^\Delta(t_{j+n}) \leq -\beta \left(\frac{r(t_j)}{r(t_{j+n})} \right)^{\frac{1}{\alpha}} \prod_{i=1}^{n-1} b_{j+i}^* < 0. \quad (3.4)$$

Consider the following impulsive dynamic inequalities

$$\begin{cases} (r(t)(y^\Delta(t))^\alpha)^\Delta \leq 0, & t \geq t_j, t \neq t_k, k = j+1, j+2, \dots, \\ y^\Delta(t_k^+) \leq b_k^* y^\Delta(t_k), & k = j+1, j+2, \dots \end{cases}$$

Let $m(t) = r(t)(y^\Delta(t))^\alpha$, then

$$\begin{cases} m^\Delta(t) \leq 0, & t \geq t_j, t \neq t_k, k = j+1, j+2, \dots, \\ m(t_k^+) \leq (b_k^*)^\alpha m(t_k), & k = j+1, j+2, \dots \end{cases}$$

Using Lemma 3.1, we obtain for $t > t_j$

$$m(t) \leq m(t_j^+) \prod_{t_j < t_k < t} (b_k^*)^\alpha,$$

i.e.

$$y^\Delta(t) \leq \left(\frac{r(t_j)}{r(t)} \right)^{\frac{1}{\alpha}} y^\Delta(t_j^+) \prod_{t_j < t_k < t} b_k^* = -\beta \left(\frac{r(t_j)}{r(t)} \right)^{\frac{1}{\alpha}} \prod_{t_j < t_k < t} b_k^*.$$

Since $y(t_k^+) \leq a_k y(t_k)$ for $k = j+1, j+2, \dots$, we obtain from Lemma 3.1

$$\begin{aligned} y(t) &\leq y(t_j^+) \prod_{t_j < t_k < t} a_k - \int_{t_j}^t \prod_{s < t_k < t} a_k \left[\beta \left(\frac{r(t_j)}{r(s)} \right)^{\frac{1}{\alpha}} \prod_{t_j < t_k < s} b_k^* \right] \Delta s \\ &= \prod_{t_j < t_k < t} a_k \left[y(t_j^+) - \beta r^{\frac{1}{\alpha}}(t_j) \int_{t_j}^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \prod_{t_j < t_k < s} \frac{b_k^*}{a_k} \Delta s \right]. \end{aligned} \quad (3.5)$$

Using (H_3) , one can find that (3.5) contradicts to $y(t) > 0$ as $t \rightarrow \infty$. Therefore, $y^\Delta(t_k) \geq 0$ for $t_k \geq T$. By the hypothesis (H_2) , one can get for any $t_k \geq T$

$$y^\Delta(t_k^+) \geq b_k^* y^\Delta(t_k) \geq 0.$$

Since $r(t)(y^\Delta(t))^\alpha$ is decreasing in $(t_k, t_{k+1}]_{\mathbb{T}}$, $t_k \geq T$, we have

$$y^\Delta(t) \geq \left(\frac{r(t_{k+1})}{r(t)} \right)^{\frac{1}{\alpha}} y^\Delta(t_{k+1}) \geq 0, \quad t \in (t_k, t_{k+1}]_{\mathbb{T}}.$$

The proof of Lemma 3.2 is complete. \square

Remark 3.1. If y is eventually negative, under the hypothesis (H_1) – (H_3) , it can be proved similarly that $y^\Delta(t_k^+) \leq 0$ and for $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $y^\Delta(t) \leq 0$ for $t_k \geq T \geq t_0$.

Lemma 3.3. Suppose that (H_1) , (H_2) hold and $y(t) > 0$, $t \geq T \geq t_0$ is a nonoscillatory solution of (1.1). If

$$(H_4) \quad \int_{t_0}^{t_1} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t + \frac{b_1^*}{a_1} \int_{t_1}^{t_2} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t + \cdots + \frac{b_1^* b_2^* \cdots b_n^*}{a_1 a_2 \cdots a_n} \int_{t_n}^{t_{n+1}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t + \cdots = \infty,$$

then $y^\Delta(t_k^+) \geq 0$ and $y^\Delta(t) \geq 0$ for $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, where $t_k \geq T$.

Proof. At first, we prove that $y^\Delta(t_k) \geq 0$ for $t_k \geq T$, otherwise, there exists some j such that $t_j \geq T$ and $y^\Delta(t_j) < 0$, similar to the proof of Lemma 3.2, we get (3.3) and (3.4) hold. Since $r(t)(y^\Delta(t))^\alpha$ is nonincreasing in $(t_j, t_{j+1}]_{\mathbb{T}}$, $t_j \geq T$, then

$$y^\Delta(t) \leq \left(\frac{r(t_j)}{r(t)} \right)^{\frac{1}{\alpha}} y^\Delta(t_j^+), \quad t \in (t_j, t_{j+1}]_{\mathbb{T}}.$$

Integrating it and using (3.3), we get

$$y(t_{j+1}) \leq y(t_j^+) + y^\Delta(t_j^+) r^{\frac{1}{\alpha}}(t_j) \int_{t_j}^{t_{j+1}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t = y(t_j^+) - \beta r^{\frac{1}{\alpha}}(t_j) \int_{t_j}^{t_{j+1}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t. \quad (3.6)$$

Similar to (3.6) and using (H_2) , (3.3) and (3.4), we get

$$\begin{aligned} y(t_{j+2}) &\leq y(t_{j+1}^+) + y^\Delta(t_{j+1}^+) r^{\frac{1}{\alpha}}(t_{j+1}) \int_{t_{j+1}}^{t_{j+2}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \\ &= g_{j+1}(y(t_{j+1})) + h_{j+1}(y^\Delta(t_{j+1})) r^{\frac{1}{\alpha}}(t_{j+1}) \int_{t_{j+1}}^{t_{j+2}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \\ &\leq a_{j+1} y(t_{j+1}) + b_{j+1}^* y^\Delta(t_{j+1}) r^{\frac{1}{\alpha}}(t_{j+1}) \int_{t_{j+1}}^{t_{j+2}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \\ &\leq a_{j+1} \left[y(t_j^+) - \beta r^{\frac{1}{\alpha}}(t_j) \int_{t_j}^{t_{j+1}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \right] - b_{j+1}^* \beta \left(\frac{r(t_j)}{r(t_{j+1})} \right)^{\frac{1}{\alpha}} r^{\frac{1}{\alpha}}(t_{j+1}) \int_{t_{j+1}}^{t_{j+2}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \\ &= a_{j+1} \left[y(t_j^+) - \beta r^{\frac{1}{\alpha}}(t_j) \left(\int_{t_j}^{t_{j+1}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t + \frac{b_{j+1}^*}{a_{j+1}} \int_{t_{j+1}}^{t_{j+2}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \right) \right]. \end{aligned}$$

By induction, it implies

$$\begin{aligned} y(t_{j+n}) &\leq a_{j+1} \cdots a_{j+n-1} \left[y(t_j^+) - \beta r^{\frac{1}{\alpha}}(t_j) \left(\int_{t_j}^{t_{j+1}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t + \frac{b_{j+1}^*}{a_{j+1}} \int_{t_{j+1}}^{t_{j+2}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \right. \right. \\ &\quad \left. \left. + \cdots + \frac{b_{j+1}^* \cdots b_{j+n-1}^*}{a_{j+1} \cdots a_{j+n-1}} \int_{t_{j+n-1}}^{t_{j+n}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t \right) \right]. \end{aligned}$$

Let $n \rightarrow \infty$ and use (H_4) , we get a contradiction to $y(t) > 0$, $t \geq T$. Therefore, $y^\Delta(t_k) \geq 0$, $(t_k \geq T)$. Similar to the proof of Lemma 3.2, we get

$$y^\Delta(t_k^+) \geq 0, \quad y^\Delta(t) \geq 0, \quad t \in (t_k, t_{k+1}]_{\mathbb{T}}, \quad t_k \geq T. \quad \square$$

Remark 3.2. If y is eventually negative, under the hypothesis (H_1) , (H_2) and (H_4) , it can be proved similarly that $y^\Delta(t_k^+) \leq 0$ and for $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $y^\Delta(t) \leq 0$ for $t_k \geq T$.

Theorem 3.1. Assume that (H_1) , (H_2) , (H_4) hold and there exists a positive integer k_0 such that $a_k^* \geq 1$ for $k \geq k_0$. If

$$\int_{t_0}^{t_1} p(t) \Delta t + \frac{1}{b_1^\alpha} \int_{t_1}^{t_2} p(t) \Delta t + \frac{1}{b_1^\alpha b_2^\alpha} \int_{t_2}^{t_3} p(t) \Delta t + \cdots + \frac{1}{b_1^\alpha b_2^\alpha \cdots b_n^\alpha} \int_{t_n}^{t_{n+1}} p(t) \Delta t + \cdots = \infty, \quad (3.7)$$

then (1.1) is oscillatory.

Proof. Suppose to the contrary that Eq. (1.1) has a nonoscillatory solution y , without loss of generality, we may assume that y is eventually positive solution of (1.1), i.e. $y(t) > 0$, $t \geq t_0$ and $k_0 = 1$. Lemma 3.3 implies $y^\Delta(t) \geq 0$, $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, \dots$. Let

$$w(t) = \frac{r(t)(y^\Delta(t))^\alpha}{\varphi(y(t))}, \quad (3.8)$$

then $w(t_k^+) \geq 0$, $k = 1, 2, \dots$ and $w(t) > 0$, $t \geq t_0$. Using (H_1) and (1.1), we get for $t \neq t_k$

$$\begin{aligned} w^\Delta(t) &= -\frac{f(t, y^\sigma(t))}{\varphi(y^\sigma(t))} - \frac{r(t)(y^\Delta(t))^\alpha}{\varphi(y(t))\varphi(y^\sigma(t))} \int_0^1 \varphi'(y(t) + h\mu(t)y^\Delta(t)) dh y^\Delta(t) \\ &\leq -p(t) - \frac{\varphi^{\frac{1}{\alpha}}(y(t))}{r^{\frac{1}{\alpha}}(t)\varphi(y^\sigma(t))} w^{\frac{\alpha+1}{\alpha}}(t) \int_0^1 \varphi'(y(t) + h\mu(t)y^\Delta(t)) dh \\ &\leq -p(t). \end{aligned} \quad (3.9)$$

Since $\varphi'(y(t)) \geq 0$ and $\varphi(y(t)) > 0$, from (H_2) and $a_k^* \geq 1$, we obtain for $k = 1, 2, \dots$

$$w(t_k^+) = \frac{r(t_k^+)(y^\Delta(t_k^+))^\alpha}{\varphi(y(t_k^+))} \leq \frac{b_k^\alpha r(t_k)(y^\Delta(t_k))^\alpha}{\varphi(a_k^* y(t_k))} \leq b_k^\alpha \frac{r(t_k)(y^\Delta(t_k))^\alpha}{\varphi(y(t_k))} = b_k^\alpha w(t_k). \quad (3.10)$$

Integrating (3.9), we get

$$w(t_1) \leq w(t_0^+) - \int_{t_0}^{t_1} p(t) \Delta t.$$

Using (3.10), we get

$$w(t_1^+) \leq b_1^\alpha w(t_1) \leq b_1^\alpha w(t_0^+) - b_1^\alpha \int_{t_0}^{t_1} p(t) \Delta t.$$

Similarly, we obtain

$$w(t_2^+) \leq b_2^\alpha w(t_2) \leq b_2^\alpha \left[w(t_1^+) - \int_{t_1}^{t_2} p(t) \Delta t \right] \leq b_1^\alpha b_2^\alpha \left[w(t_0^+) - \int_{t_0}^{t_1} p(t) \Delta t - \frac{1}{b_1^\alpha} \int_{t_1}^{t_2} p(t) \Delta t \right].$$

By induction, for any positive integer n , we have

$$w(t_n^+) \leq b_1^\alpha b_2^\alpha \cdots b_n^\alpha \left[w(t_0^+) - \int_{t_0}^{t_1} p(t) \Delta t - \frac{1}{b_1^\alpha} \int_{t_1}^{t_2} p(t) \Delta t - \cdots - \frac{1}{b_1^\alpha \cdots b_{n-1}^\alpha} \int_{t_{n-1}}^{t_n} p(t) \Delta t \right].$$

Using (3.7) and $b_k > 0$, $k = 1, 2, \dots$, we obtain $w(t_n^+) \rightarrow -\infty$ as $n \rightarrow \infty$, which contradicts to $w(t_n^+) \geq 0$. Then every solution of Eq. (1.1) is oscillatory. \square

Theorem 3.2. Assume that (H_1) – (H_3) hold and there exists a positive integer k_0 such that $a_k^* \geq 1$ for $k \geq k_0$. If

$$\int_{t_0}^{\infty} \prod_{t_0 < t_k < t} \frac{1}{b_k^\alpha} p(t) \Delta t = \infty, \quad (3.11)$$

then (1.1) is oscillatory.

Proof. Similar to the proof of Theorem 3.1, we may assume that y is eventually positive solution of (1.1), i.e. $y(t) > 0$, $t \geq t_0$ and $k_0 = 1$. From Lemma 3.2, we have $y^\Delta(t) \geq 0$, $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, \dots$. Define $w(t)$ as in (3.8), we get (3.9) and (3.10) hold. Applying Lemma 3.1, we obtain from (3.9) and (3.10)

$$w(t) \leq w(t_0) \prod_{t_0 < t_k < t} b_k^\alpha - \int_{t_0}^t \prod_{s < t_k < s} b_k^\alpha p(s) \Delta s = \prod_{t_0 < t_k < t} b_k^\alpha \left[w(t_0) - \int_{t_0}^t \prod_{t_0 < t_k < s} \frac{1}{b_k^\alpha} p(s) \Delta s \right]. \quad (3.12)$$

In view of (3.11) and (3.12), we get a contradiction as $t \rightarrow \infty$. Then every solution of (1.1) is oscillatory. \square

Theorem 3.3. Assume that (H_1) , (H_2) , (H_4) hold and $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. If

$$\int_{t_0}^{t_1} p(t) \Delta t + \frac{\varphi(a_1^*)}{b_1^\alpha} \int_{t_1}^{t_2} p(t) \Delta t + \frac{\varphi(a_1^*)\varphi(a_2^*)}{b_1^\alpha b_2^\alpha} \int_{t_2}^{t_3} p(t) \Delta t + \cdots + \frac{\varphi(a_1^*) \cdots \varphi(a_n^*)}{b_1^\alpha \cdots b_n^\alpha} \int_{t_n}^{t_{n+1}} p(t) \Delta t + \cdots = \infty, \quad (3.13)$$

then (1.1) is oscillatory.

Proof. As before, suppose $y(t) > 0$, $t \geq t_0$ be a nonoscillatory solution of (1.1), Lemma 3.2 yields $y^\Delta(t) \geq 0$, $t \geq t_0$, define $w(t)$ as in (3.8), we get $w(t) \geq 0$, $t \geq t_0$, $w(t_k^+) \geq 0$, $k = 1, 2, \dots$, (3.9) holds for $t \neq t_k$ and

$$w(t_k^+) = \frac{r(t_k^+)(y^\Delta(t_k^+))^\alpha}{\varphi(y(t_k^+))} \leq \frac{b_k^\alpha r(t_k)(y^\Delta(t_k))^\alpha}{\varphi(a_k^* y(t_k))} \leq \frac{b_k^\alpha r(t_k)(y^\Delta(t_k))^\alpha}{\varphi(a_k^*)\varphi(y(t_k))} = \frac{b_k^\alpha}{\varphi(a_k^*)} w(t_k). \quad (3.14)$$

Similar to the proof of Theorem 3.1, by induction, for any positive integer n , one get

$$w(t_n^+) \leq \frac{b_1^\alpha \cdots b_n^\alpha}{\varphi(a_1^*) \cdots \varphi(a_n^*)} \left[w(t_0^+) - \int_{t_0}^{t_1} p(t) \Delta t - \frac{\varphi(a_1^*)}{b_1^\alpha} \int_{t_1}^{t_2} p(t) \Delta t - \cdots - \frac{\varphi(a_1^*) \cdots \varphi(a_{n-1}^*)}{b_1^\alpha \cdots b_{n-1}^\alpha} \int_{t_{n-1}}^{t_n} p(t) \Delta t \right].$$

Let $n \rightarrow \infty$ and use (3.13), we obtain the desired contradiction. \square

Theorem 3.4. Assume that (H_1) – (H_3) hold and $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. If

$$\int_{t_0}^{\infty} \prod_{t_0 < t_k < t} \frac{\varphi(a_k^*)}{b_k^\alpha} p(t) \Delta t = \infty, \quad (3.15)$$

then (1.1) is oscillatory.

Proof. Similar to the proof of Theorem 3.3, using Lemma 3.2 to (3.9) and (3.14), we obtain the desired contradiction. \square

Remark 3.3. When $r(t) = 1$, $\alpha = 1$, then Theorems 3.1 and 3.3 reduce to Theorems 2.3 and 2.4 of Huang [12], and Theorems 3.2 and 3.4 reduce to Theorems 3.1 and 3.2 of Huang [13].

Remark 3.4. When $r(t) = 1$, $\alpha = 1$ and the time scale $\mathbb{T} = \mathbb{R}$, our Theorems 3.1–3.4 reduce to the main results of Chen [14] and Peng [15] without delay. When the time scale $\mathbb{T} = \mathbb{N}$, we can obtain some interesting oscillation criteria for second order nonlinear difference equations with impulses.

In the following, we will use the hypothesis

$$(H_5) \int_{\pm\epsilon}^{\pm\infty} \frac{\Delta u}{\varphi(\frac{1}{\alpha}(u))} < \infty, \text{ for any } \epsilon > 0,$$

where $\int_{\pm\epsilon}^{\pm\infty} \frac{\Delta u}{\varphi(\frac{1}{\alpha}(u))} < \infty$ denotes $\int_{\epsilon}^{\infty} \frac{\Delta u}{\varphi(\frac{1}{\alpha}(u))} < \infty$ and $\int_{-\infty}^{-\epsilon} \frac{\Delta u}{\varphi(\frac{1}{\alpha}(u))} < \infty$.

Theorem 3.5. Assume that (H_1) – (H_3) , (H_5) hold and there exists a positive integer k_0 such that $a_k^* \geq 1$ for $k \geq k_0$. If

$$\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\int_s^{\infty} \prod_{s < t_k < \tau} \frac{1}{b_k^\alpha} p(\tau) \Delta \tau \right)^{\frac{1}{\alpha}} \Delta s = \infty, \quad (3.16)$$

then (1.1) is oscillatory.

Proof. Assume $y(t) > 0$, $t \geq t_0$ be a nonoscillatory solution of (1.1) and $k_0 = 1$, Lemma 3.2 shows that $y^\Delta(t_k^+) \geq 0$, $k = 1, 2, \dots$ and $y^\Delta(t) \geq 0$, $t \geq t_0$. Since $a_k^* \geq 1$, $k = 1, 2, \dots$, we get

$$y(t_0^+) \leq y(t_1) \leq y(t_1^+) \leq y(t_2) \leq y(t_2^+) \leq \cdots,$$

its easy to see that $y(t)$ is nondecreasing in $[t_0, \infty)_{\mathbb{T}}$, (1.1) yields

$$\begin{cases} (r(t)(y^\Delta(t))^\alpha)^\Delta = -f(t, y^\sigma(t)) \leq -p(t)\varphi(y^\sigma(t)), & t \neq t_k, k = 1, 2, \dots \\ y^\Delta(t_k^+) \leq b_k y^\Delta(t_k), & k = 1, 2, \dots \end{cases}$$

Let $m(t) = r(t)(y^\Delta(t))^\alpha$, then

$$\begin{cases} m^\Delta(t) \leq -p(t)\varphi(y^\sigma(t)), & t \neq t_k, k = 1, 2, \dots \\ m(t_k^+) \leq b_k^\alpha m(t_k), & k = 1, 2, \dots \end{cases}$$

It follows from Lemma 3.1 that

$$m(t) \leq m(s) \prod_{s < t_k < t} b_k^\alpha - \int_s^t \prod_{\tau < t_k < t} b_k^\alpha p(\tau) \varphi(y^\sigma(\tau)) \Delta \tau, \quad t_0 \leq s \leq t,$$

i.e.

$$r(t)(y^\Delta(t))^\alpha \leq r(s)(y^\Delta(s))^\alpha \prod_{s < t_k < t} b_k^\alpha - \int_s^t \prod_{\tau < t_k < t} b_k^\alpha p(\tau) \varphi(y^\sigma(\tau)) \Delta \tau, \quad t_0 \leq s \leq t.$$

This implies for $t_0 \leq s \leq t$

$$y^\Delta(s) \geq \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\int_s^t \prod_{s < t_k < \tau} \frac{1}{b_k^\alpha} p(\tau) \varphi(y^\sigma(\tau)) \Delta \tau \right)^{\frac{1}{\alpha}}.$$

Since $\varphi(y) > 0$ ($y \neq 0$) and $\varphi(y)$ is nondecreasing, we get

$$\frac{y^\Delta(s)}{\varphi^{\frac{1}{\alpha}}(y(s))} \geq \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\int_s^t \prod_{s < t_k < \tau} \frac{1}{b_k^\alpha} p(\tau) \frac{\varphi(y^\sigma(\tau))}{\varphi(y(s))} \Delta \tau \right)^{\frac{1}{\alpha}} \geq \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\int_s^t \prod_{s < t_k < \tau} \frac{1}{b_k^\alpha} p(\tau) \Delta \tau \right)^{\frac{1}{\alpha}},$$

for $s \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, \dots$. Then

$$\int_{t_k}^{t_{k+1}} \frac{y^\Delta(s)}{\varphi^{\frac{1}{\alpha}}(y(s))} \Delta s = \int_{y(t_k^+)}^{y(t_{k+1})} \frac{\Delta \tau}{\varphi^{\frac{1}{\alpha}}(\tau)}.$$

It implies that

$$\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\lim_{t \rightarrow \infty} \int_s^t \prod_{s < t_k < \tau} \frac{1}{b_k^\alpha} p(\tau) \Delta \tau \right)^{\frac{1}{\alpha}} \Delta s \leq \sum_{k=0}^{\infty} \int_{y(t_k^+)}^{y(t_{k+1})} \frac{\Delta \tau}{\varphi^{\frac{1}{\alpha}}(\tau)} \leq \int_{y(t_0^+)}^{\infty} \frac{\Delta \tau}{\varphi^{\frac{1}{\alpha}}(\tau)},$$

which contradicts to (3.16), then every solution of (1.1) is oscillatory. \square

Theorem 3.6. Suppose that (H_1) – (H_3) , (H_5) hold and $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. If

$$\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\int_s^{\infty} \prod_{s < t_k < \tau} \frac{\varphi(a_k^*)}{b_k^\alpha} p(\tau) \Delta \tau \right)^{\frac{1}{\alpha}} \Delta s = \infty, \quad (3.17)$$

Then (1.1) is oscillatory.

Proof. As before, we may assume that $y(t) > 0$, $t \geq t_0$ be a nonoscillatory solution of (1.1). Lemma 3.2 yields $y^\Delta(t) \geq 0$, $t \geq t_0$. Define $w(t)$ as in (3.8) and we get (3.9) hold for $t \neq t_k$ and (3.14) hold. Using Lemma 3.1 to (3.9) and (3.14), we get

$$w(t) \leq w(s) \prod_{s < t_k < t} \frac{b_k^\alpha}{\varphi(a_k^*)} - \int_s^t \prod_{\tau < t_k < t} \frac{b_k^\alpha}{\varphi(a_k^*)} p(\tau) \Delta \tau, \quad t_0 \leq s \leq t.$$

It yields that

$$w(s) \geq \int_s^t \prod_{s < t_k < \tau} \frac{\varphi(a_k^*)}{b_k^\alpha} p(\tau) \Delta \tau,$$

i.e.

$$\frac{y^\Delta(s)}{\varphi^{\frac{1}{\alpha}}(y(s))} \geq \left(\frac{1}{r(s)} \int_s^t \prod_{s < t_k < \tau} \frac{\varphi(a_k^*)}{b_k^\alpha} p(\tau) \Delta \tau \right)^{\frac{1}{\alpha}},$$

for $s \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k = 1, 2, \dots$. Then

$$\int_{t_k}^{t_{k+1}} \frac{y^\Delta(s)}{\varphi^{\frac{1}{\alpha}}(y(s))} \Delta s = \int_{y(t_k^+)}^{y(t_{k+1})} \frac{\Delta \tau}{\varphi^{\frac{1}{\alpha}}(\tau)}.$$

Then

$$\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\lim_{t \rightarrow \infty} \int_s^t \prod_{s < t_k < \tau} \frac{\varphi(a_k^*)}{b_k^\alpha} p(\tau) \Delta \tau \right)^{\frac{1}{\alpha}} \Delta s \leq \sum_{k=0}^{\infty} \int_{y(t_k^+)}^{y(t_{k+1})} \frac{\Delta \tau}{\varphi^{\frac{1}{\alpha}}(\tau)} \leq \int_{y(t_0^+)}^{\infty} \frac{\Delta \tau}{\varphi^{\frac{1}{\alpha}}(\tau)},$$

which is in contradiction with (3.17), then every solutions of (1.1) is oscillatory. \square

Remark 3.5. Using the hypothesis (H₄) instead (H₃) in the Theorems 3.5 and 3.6, we can establish some oscillation criteria for Eq. (1.1).

From Theorems 3.1–3.6, we have the following corollaries.

Corollary 3.1. Suppose that (H₁)–(H₃) hold and there exists a positive integer k_0 such that $a_k^* \geq 1$, $b_k \leq 1$ for $k \geq k_0$. If $\int_{t_0}^{\infty} p(t) \Delta t = \infty$, then (1.1) is oscillatory.

Proof. Without loss of generality, let $k_0 = 1$, by $b_k \leq 1$, we get $\frac{1}{b_k^\alpha} \geq 1$, therefore

$$\int_{t_0}^t \prod_{t_0 < t_k < s} \frac{1}{b_k^\alpha} p(s) \Delta s \geq \int_{t_0}^t p(s) \Delta s.$$

Let $t \rightarrow \infty$ and using $\int_{t_0}^{\infty} p(t) \Delta t = \infty$, we obtain from Theorem 3.2 that (1.1) is oscillatory. \square

Corollary 3.2. Suppose that (H₁)–(H₃), (H₅) hold and there exists a positive integer k_0 such that $a_k^* \geq 1$, $b_k \leq 1$ for $k \geq k_0$. If

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\int_s^{\infty} p(t) \Delta t \right)^{\frac{1}{\alpha}} \Delta s = \infty,$$

then (1.1) is oscillatory.

Using Theorem 3.5, the proof of Corollary 3.2 is similar to the proof of Corollary 3.1.

Corollary 3.3. Suppose that (H₁), (H₂), (H₄) hold and there exist a positive integer k_0 and a constant $\gamma > 0$ such that

$$a_k^* \geq 1, \quad \frac{1}{b_k^\alpha} \geq \left(\frac{t_{k+1}}{t_k} \right)^\gamma, \quad \text{for } k \geq k_0. \quad (3.18)$$

If

$$\int_{t_0}^{\infty} t^\gamma p(t) \Delta t = \infty. \quad (3.19)$$

Then (1.1) is oscillatory.

Proof. Without loss of generality, let $k_0 = 1$, then

$$\begin{aligned} & \int_{t_0}^{t_1} p(t) \Delta t + \frac{1}{b_1^\alpha} \int_{t_1}^{t_2} p(t) \Delta t + \cdots + \frac{1}{b_1^\alpha b_2^\alpha \cdots b_n^\alpha} \int_{t_n}^{t_{n+1}} p(t) \Delta t \\ & \geq \frac{1}{t_1^\gamma} \left[\int_{t_1}^{t_2} t_2^\gamma p(t) \Delta t + \int_{t_2}^{t_3} t_3^\gamma p(t) \Delta t + \cdots + \int_{t_n}^{t_{n+1}} t_{n+1}^\gamma p(t) \Delta t \right] \\ & \geq \frac{1}{t_1^\gamma} \left[\int_{t_1}^{t_2} s^\gamma p(s) \Delta s + \int_{t_2}^{t_3} s^\gamma p(s) \Delta s + \cdots + \int_{t_n}^{t_{n+1}} s^\gamma p(s) \Delta s \right] \\ & = \frac{1}{t_1^\gamma} \int_{t_1}^{t_{n+1}} s^\gamma p(s) \Delta s. \end{aligned} \quad (3.20)$$

Let $n \rightarrow \infty$ and use (3.19) and (3.20) yields (3.7) holds. According to Theorem 3.1, we obtain (1.1) is oscillatory. \square

Corollary 3.4. Assume that (H₁), (H₂), (H₄) hold and $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. Suppose there exist a positive integer k_0 and a constant $\gamma > 0$ such that

$$\frac{\varphi(a_k^*)}{b_k^\alpha} \geq \left(\frac{t_{k+1}}{t_k} \right)^\gamma, \quad \text{for } k \geq k_0.$$

If $\int_{t_0}^{\infty} t^\gamma p(t) \Delta t = \infty$, then (1.1) is oscillatory.

Corollary 3.4 can be deduced from Theorem 3.3. Its proof is similar to that of Corollary 3.3, here we omit it.

4. Examples

Example 4.1. Consider the following second order impulsive dynamic equation

$$\begin{cases} \left(\frac{1}{t} (y^\Delta(t))^{\frac{1}{3}} \right)^\Delta + t^\theta y^5(\sigma(t)) = 0, & t \geq \frac{1}{2}, t \neq k, k = 1, 2, \dots, \\ y(k^+) = y(k), & y^\Delta(k^+) = \frac{k}{k+1} y^\Delta(k), \quad k = 1, 2, \dots, \\ y\left(\frac{1}{2}\right) = y_0, & y^\Delta\left(\frac{1}{2}\right) = y_0^\Delta, \end{cases} \quad (4.1)$$

where $\theta \geq -\frac{1}{3}$.

Since $r(t) = \frac{1}{t}$, $\alpha = \frac{1}{3}$, $a_k = a_k^* = 1$, $b_k = b_k^* = \frac{k}{k+1}$, $p(t) = t^\theta$, $t_k = k$, and $\varphi(y) = y^5$, it is obvious that the hypothesis (H_1) , (H_2) , (H_4) hold. $ab > 0$ implies $\varphi(ab) = \varphi(a)\varphi(b)$. Let $k_0 = 1$, $\gamma = \frac{1}{3} > 0$, then

$$\frac{\varphi(a_k^*)}{b_k^\alpha} = \frac{1}{\left(\frac{k}{k+1}\right)^{\frac{1}{3}}} = \left(\frac{k+1}{k}\right)^{\frac{1}{3}} = \left(\frac{t_{k+1}}{t_k}\right)^{\frac{1}{3}} = \left(\frac{t_{k+1}}{t_k}\right)^\gamma,$$

and

$$\int_1^\infty t^\gamma p(t) \Delta t = \int_1^\infty t^{\frac{1}{3}} t^\theta \Delta t \geq \int_1^\infty t^{\frac{1}{3}} t^{-\frac{1}{3}} \Delta t = \infty.$$

Corollary 3.4 implies Eq. (4.1) is oscillatory.

Example 4.2. Consider the following second order impulsive dynamic equation

$$\begin{cases} y^{\Delta\Delta}(t) + \frac{1}{t\sigma^2(t)} y^\theta(\sigma(t)) = 0, & t \geq 1, t \neq k, k = 1, 2, \dots, \\ y(k^+) = \frac{k+1}{k} y(k), & y^\Delta(k^+) = y^\Delta(k), \quad k = 1, 2, \dots, \\ y(1) = y_0, & y^\Delta(1) = y_0^\Delta, \end{cases} \quad (4.2)$$

where $\theta \geq 3$ is a odd integer and $\mu(t) \leq ct$, where c is a positive constant.

Since $r(t) = 1$, $\alpha = 1$, $a_k = a_k^* = \frac{k+1}{k}$, $b_k = b_k^* = 1$, $p(t) = \frac{1}{t\sigma^2(t)}$, $t_k = k$ and $\varphi(y) = y^\theta$. It is easy to see that (H_1) , (H_2) , (H_4) hold. Let $k_0 = 1$, $\gamma = 3$, hence

$$\frac{\varphi(a_k^*)}{b_k} = \left(\frac{k+1}{k}\right)^\theta = \left(\frac{t_{k+1}}{t_k}\right)^\theta \geq \left(\frac{t_{k+1}}{t_k}\right)^3,$$

and

$$\int_1^\infty t^\gamma p(t) \Delta t = \int_1^\infty t^3 \frac{1}{t\sigma^2(t)} \Delta t = \int_1^\infty \left(\frac{t}{\sigma(t)}\right)^2 \Delta t.$$

Since $\mu(t) \leq ct$, we get

$$\frac{t}{\sigma(t)} = \frac{t}{t + \mu(t)} \geq \frac{1}{1 + c},$$

hence

$$\int_1^\infty \left(\frac{t}{\sigma(t)}\right)^2 \Delta t \geq \frac{1}{(1+c)^2} \int_1^\infty \Delta t = \infty.$$

By Corollary 3.4, we obtain that (4.2) is oscillatory. But by [20] we know that the dynamic equation $y^{\Delta\Delta}(t) + \frac{1}{t\sigma^2(t)} y^\theta(\sigma(t)) = 0$ is nonoscillatory.

In the above example, it is interesting to note that the dynamic equation without impulses is nonoscillatory, but when some impulses are added to it, it becomes oscillatory. Therefore, this example shows that impulses play an important part in the oscillations of dynamic equations on time scales.

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